

A simple model for the microscopic effective pairing interaction

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Abstract

The microscopic effective pairing interaction in the 1S_0 -channel is investigated for two different values of the chemical potential μ starting from the separable form of the Paris NN-potential. It is shown that, within a high accuracy, this effective interaction can be approximated by the off-shell free T -matrix taken at the negative energy $E = 2\mu$.

Recently [1] the microscopic effective pairing interaction $\mathcal{V}_{\text{eff}}^{\text{p}}$ in the 1S_0 -channel was found by solving the Bethe-Goldstone equation for semi-infinite nuclear matter without any form of local approximation. The separable representation [2, 3] of the Paris potential [4] was used which simplifies the problem significantly and makes it possible such direct solution for non-uniform systems with the use of the mixed coordinate-momentum representation. However, the procedure turned out to be very cumbersome and the effective interaction obtained has rather complicated form not convenient for applications. In this paper we show that $\mathcal{V}_{\text{eff}}^{\text{p}}$, with a good accuracy, can be approximated by the free T -matrix taken at the energy $E = 2\mu$. This quantity is much simpler than $\mathcal{V}_{\text{eff}}^{\text{p}}$ and can be easily found for a fixed value of μ .

The many-body theory form of the gap equation [5, 6]

$$\Delta = \mathcal{V}GG^s\Delta \quad (1)$$

is used in [1] which explicitly takes into consideration the particle-particle propagator $A^s = GG^s$ in the superfluid system. Within the Bethe-Brueckner approach, the irreducible particle-particle interaction block \mathcal{V} is approximated by the free NN -potential. The effective interaction is associated with the splitting of the complete Hilbert space S into two domains, the model subspace S_0 and the complementary one S' . As a result, the two-particle propagator is represented as a sum $A^s = A_0^s + A'$. It is supposed that the superfluid effects can be neglected in the S' -subspace and therefore the upperscript “s” is omitted in the second term.

The gap equation (1) can be rewritten in the model subspace:

$$\Delta = \mathcal{V}_{\text{eff}}^{\text{p}}A_0^s\Delta, \quad (2)$$

where $\mathcal{V}_{\text{eff}}^{\text{p}}$ obeys the following equation:

$$\mathcal{V}_{\text{eff}}^{\text{p}} = \mathcal{V} + \mathcal{V}A'\mathcal{V}_{\text{eff}}^{\text{p}}. \quad (3)$$

In [1] the model space S_0 is defined in such a way that it involves all the two-particle states (λ, λ') with the single-particle energies $\varepsilon_\lambda, \varepsilon_{\lambda'}$ both negative. In this case, the complementary subspace S' involves all the two-particle states with positive energies $\varepsilon_\lambda, \varepsilon_{\lambda'}$ and two-particle states with one energy positive and the second one negative (but greater than μ).

The separable form [2, 3] of the Paris potential is as follows:

$$\mathcal{V}(k, k') = \sum_{ij} \lambda_{ij} g_i(k^2) g_j(k'^2), \quad (4)$$

where form factors $g_i(k^2)$, ($i = 1, 2, 3$) are rational functions. This choice was adopted earlier in the Brueckner-type calculations for infinite nuclear matter [7, 8].

It should be noted that the original normalization [2, 3] of the expansion (4) was changed in [1] in such a way that the identity $g_i(0) = 1$ holds true. Then the absolute values of the λ_{ij} -coefficients give direct information on the strength of the corresponding terms of the force. Their values (in $\text{MeV}\cdot\text{fm}^3$) are as follows: $\lambda_{11} = -3659$, $\lambda_{12} = 2169$, $\lambda_{22} = -1485$ and $\lambda_{13} = -23.6$, $\lambda_{23} = 57.6$, $\lambda_{33} = 17.2$. As it is seen, the strengths of all the components containing only the indices $i = 1, 2$ are much stronger than those with the index $i = 3$. Therefore the latter will not be considered for a qualitative analysis. Of course, in the calculations all the terms λ_{ik} are considered.

The separable representation of the NN -potential leads to a similar form of $\mathcal{V}_{\text{eff}}^p$ which, in the notation of [1] is as follows:

$$\mathcal{V}_{\text{eff}}^p(k_{\perp}^2, k_{\perp}'^2; x_1, x_2, x_3, x_4; E) = \sum_{ij} \Lambda_{ij}(X_{12}, X_{34}; E) g_i(k_{\perp}^2, x_{12}) g_j(k_{\perp}'^2, x_{34}). \quad (5)$$

Here the center-of-mass and relative coordinates in the x -direction are introduced ($X_{12} = (x_1 + x_2)/2$, $x_{12} = x_1 - x_2$, etc.), and $g_i(k_{\perp}^2, x)$ stands for the inverse Fourier transform of the form factors $g_i(k_{\perp}^2 + k_x^2)$ in the x -direction.

The coefficients Λ_{ij} obey the set of integral equations:

$$\begin{aligned} \Lambda_{ij}(X_{12}, X_{34}; E) &= \lambda_{ij} \delta(X_{12} - X_{34}) + \\ &+ \sum_{lm} \lambda_{il} \int dX_{56} B_{lm}(X_{12}, X_{56}; E) \Lambda_{mj}(X_{56}, X_{34}; E), \end{aligned} \quad (6)$$

where B_{lm} are given by integrals of the propagator A' with two form factors. Their explicit form is as follows:

$$B_{lm}(X_{12}, X_{34}; E) = \sum'_{nn'} \int \frac{d^2 \mathbf{k}_{\perp}}{(2\pi)^2} \frac{G_{nn'}^l(k_{\perp}^2, X_{12}) G_{n'n}^{m*}(k_{\perp}^2, X_{34})}{E - \varepsilon_{\lambda} - \varepsilon_{\lambda'}}, \quad (7)$$

$$G_{n,n'}^l(k_{\perp}^2, X_{12}) = \int dx_{12} g_l(k_{\perp}^2, x_{12}) y_n(X_{12} + x_{12}/2) y_{n'}(X_{12} - x_{12}/2), \quad (8)$$

where $\lambda = (n, \mathbf{k}_{\perp})$, $\varepsilon_{\lambda} = \varepsilon_n + k_{\perp}^2/2m$, ε_n , y_n stand for the energies and wave functions of the one-dimensional Schrödinger equation, respectively. The prime in the sum of eq. (7) means that the summation is carried out over (λ, λ') which are not included in the model space. It is convenient to put in this sum $\varepsilon_{\lambda} < \varepsilon_{\lambda'}$ (multiplying the result by the factor 2). Then the sum contains all the states with $\varepsilon_{\lambda} > \mu$, $\varepsilon_{\lambda'} > 0$.

To overcome a problem of a slow convergence of the integrals of eq. (7), a renormalization can be made in terms of the free T -matrix:

$$T = \mathcal{V} + \mathcal{V} A^0 T, \quad (9)$$

where A^0 is the propagator of two free particles.

Eqs. (6), (7) hold true for the T -matrix (with the substitution $T \rightarrow \Lambda, B^0 \rightarrow B$), but now the coefficients $T_{ij}(X, X'; E), B_{ij}^0(X, X'; E)$ of the separable expansion depend only on the difference $t = X - X'$ of the CM coordinates:

$$T_{ij}(t; E) = \lambda_{ij}\delta(t) + \sum_{lm} \lambda_{il} \int dt' B_{lm}(t - t'; E) \Lambda_{mj}(t'; E), \quad (10)$$

The renormalized equation for the effective interaction in a compact form reads:

$$\Lambda_{ij} = T_{ij} + \sum_{lm} T_{il} (B_{lm} - B_{lm}^0) T_{mj} \quad (11)$$

The kernel of this equation converges much faster than the original one. Of course, the problem of the slow convergence at large momenta does not disappear. It passes to eq. (10), but in this case the difficulty can be overcome much more easily. Indeed, we are dealing now with the one-dimensional vector $T_{ij}(t)$ instead of the two-dimensional matrix $\Lambda_{ij}(X, X')$. It is convenient first to find the T -matrix in the momentum representation by solving the following set of equations:

$$T_{ij}(P_x; E) = \lambda_{ij} + \sum_{lm} \lambda_{il} B_{lm}^0(P_x; E) T_{mj}(P_x; E), \quad (12)$$

where

$$B_{lm}^0(P_x; E) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{g_l(k^2) g_m(k^2)}{E - P_x^2/4m - k^2/m}. \quad (13)$$

Then $T_{ij}(t)$ can be found from $T_{ij}(P_x)$ with the help of the inverse Fourier transformation:

$$T_{ij}(t; E) = \int_{-\infty}^{\infty} \frac{dP_x}{2\pi} T_{ij}(P_x; E) \exp(-iP_x t). \quad (14)$$

The form factors g_i in eq. (13) are rational functions on k^2 [2, 3], namely, combinations of the Yukawa function and their derivatives with different masses β_{in} ($n = 1, \dots, 4$). This integral can be evaluated analytically, but such calculation is very cumbersome because of a huge number ($\simeq 70$) of particular terms appearing in the integrand. We prefer, following [1], to integrate it numerically, with the cut-off momentum $k_c = 60 \text{ fm}^{-1}$ which guarantees an accuracy better than 1%.

The Fourier integral (14), after separating the constant term λ_{ij} , was calculated in [1] by direct integration along the real P_x -axis. This method works well at small t , but for $t > (2 \div 3) \text{ fm}$ the integrand contains rapidly oscillating factors multiplied by a slowly falling function $(T_{ij}(P_x^2) - \lambda_{ij}) \simeq 1/P_x^2$, which makes the convergence very poor. To obtain a reasonable accuracy at $t = (4 \div 5) \text{ fm}$, the large value of the cut-off momentum $P_x^c = 3000 \text{ fm}^{-1}$ was taken in [1] with very small integration step. It is much better, in accordance with the general recipe of integrating rapidly oscillating

functions, to integrate eq. (14) in the complex plane of P_x . The integration contour C can be closed in the upper half-plane and deformed to a form convenient for numerical integrating (Fig. 1). As it can be readily shown, in the case of rational form factors g_i all the singularities of $T_{ij}(P_x; E)$ are the poles (simple and multiple) located on the imaginary axis (symmetrically to the origin). Their position $P_x^\alpha = i\gamma_\alpha$ is defined by various combinations of the masses β_{in} and the μ -dependent parameter $\gamma_0 = \sqrt{-8m\mu}$. The contour C surrounds all these poles, each one yielding a falling exponent $\exp(-\gamma_\alpha|t|)$. Besides, the procedure of numerical integration of eq. (13) produces some additional “false” poles depending on k_c . Though their contribution is negligible, the contour surrounds them also for sake of consistency. In practice, we used the contour C with values of parameters $a = 2 \text{ fm}^{-1}$, $b = 120 \text{ fm}^{-1}$. In this case it is placed far enough from all the poles and the integral can be calculated without numerical problems yielding the correct result for any value of t .

We describe the procedure of calculating the off-shell T -matrix in detail because, as will be shown, the difference $B - B_0$ in eq. (11) turns out to be rather small and the solution of this equation with a good accuracy coincides with the T -matrix. Therefore the evaluation of this quantity is of primary importance.

In this paper, we go from the semi-infinite system to a more realistic geometry of a finite slab within the Saxon-Woods potential well. The parameters are chosen in such a way to reproduce qualitatively those of heavy nuclei in the lead region: the width of the slab is $2L = 16 \text{ fm}$, the depth of the well $V_0 = -50 \text{ MeV}$ and the diffuseness parameter $d = 0.65 \text{ fm}$. In the slab case, all the above equations are valid with substitution of the slab wave functions and energies in eqs. (7), (8). Instead of direct solution of these equations, the Local Potential Approximation (LPA) is used which was previously proved to be a reliable approximation for semi-infinite nuclear matter [1]. Within the LPA, the exact values of $B_{lm}(X_1, X_2; E)$ are replaced by the set of those $B_{lm}^{\text{inf}}(t, E; V[X])$ for infinite nuclear matter put in the potential well $V[X]$. Here $X = (X_1 + X_2)/2$ is the average value of the two CM coordinates. As it is shown in [1], comparing exact solution for semi-infinite matter with the LPA prediction, the latter works with an accuracy of few percent even in the surface region. It is natural to suppose that in the slab case the LPA works also sufficiently well. The LPA procedure is as follows. At a fixed value of the chemical potential μ , it is necessary, first, to calculate the set of vectors $B_{lm}^{\text{inf}}(t, E = 2\mu; V_i)$ ($V_i = \delta V \cdot (i-1)$), and then, for every value of (X, t) , to find $B_{lm}^{\text{LPA}}(X_1, X_2)$ by interpolating values of $B_{lm}^{\text{inf}}(t; V_i)$ with nearest to $V(X)$ values of V_i .

Let us first take $\mu = -8 \text{ MeV}$. Shown in Fig. 2 is the difference $(B_{lm}^{\text{inf}}(V) - B_{lm}^0)$ for three main components lm , in comparison with the free propagators B_{lm}^0 , for the case of the maximum value of the potential depth $V = 50 \text{ MeV}$. Of course, for smaller values of V this difference is even less. To analyse the origin of this smallness, let us separate the sum of eq. (7) into two parts, the first one with both

positive energies and the second one with one negative energy, and compare each of them with the corresponding sum for the free propagator. It should be stressed that within the LPA eq. (7) is very similar to that for B_{lm}^0 because the plane waves in the constant potential $V_0=V(X)$ stand for y_n in this case. Every term of the first sum corresponds directly to the one of B_{lm}^0 with the same energy denominator. It can be easily seen that the numerator of the in-matter term is less than that of the free one because the matrix elements of the form factors are calculated for bigger values of momenta ($q = \sqrt{p^2 + 2mV}$ in the in-matter case in comparison with p in the free one). Since all the form factors $g_i(p^2)$ fall with p , this part of $B_{lm}^{\text{inf}}(V)$ (in absolute value) is less than B_{lm}^0 (at 20 ÷ 30 %). The second part of the sum under consideration has no analogue in the free propagator. Though it contains an essentially smaller phase space than the first one, its value turns out to be significant due to small value of the energy denominators. Its contribution to the absolute value of $B_{lm}^{\text{inf}}(V)$ is even larger than the difference of the first term of B_{lm} and B_{lm}^0 , so the absolute value of B_{lm} exceeds that of B_{lm}^0 , but only a bit. The size of the difference is about 10%.

Fig. 3 shows similar comparison of the effective interaction $\mathcal{V}_{\text{eff}}^{\text{inf}} [V=50 \text{ MeV}]$ and the free T-matrix. Of course, the δ -terms of both amplitudes (see eqs. (6), (10)) are extracted. For more descriptive comparison of the “exact” LPA effective interaction and the free T -matrix we display in Fig. 4 their zero-order moments:

$$\bar{\Lambda}_{ij}(X) = \int dt \Lambda_{ij}(X-t/2, X+t/2), \quad (15)$$

and the same for \bar{T}_{ij} . As it is seen, the difference between these averages is again very small for the 11-component, but for other components it is greater, namely $\simeq 20\%$ for the 12-component and $\simeq 30\%$ for the 22-one. However, these two deviations are of the opposite sign and, as we show just below, almost compensate each other in the sum at values of k^2, k'^2 which are important for the pairing problem.

It can be demonstrated by analysing the localized form of the effective interaction which was introduced in [1]:

$$\bar{\mathcal{V}}_{\text{eff}}^F(X) = \sum_{ij} \bar{\Lambda}_{ij}(X) g_i(k_F^2(X)) g_j(k_F^2(X)), \quad (16)$$

where $k_F^2(X)=2m(\mu-U(X))$ if $\mu-U(X) > 0$ and $k_F^2(X)=0$ in the opposite case. In [1], it is argued that, for states in the Fermi surface vicinity, the effective interaction can be approximately replaced by $\bar{\mathcal{V}}_{\text{eff}}^F(X)$. It is drawn in Fig. 5, together with \bar{T}^F which is defined in a similar way. One sees that the compensation discussed above, indeed, occurs and the difference between the two curves is practically negligible.

To imitate the situation in the vicinity of drip-lines, all the calculations were repeated for $\mu = -4 \text{ MeV}$. Results are completely similar to those for $\mu = -8 \text{ MeV}$.

They are shown in Fig. 6 for the B_{11} component and in Fig. 5, for the local form of the effective interaction $\bar{\mathcal{V}}_{\text{eff}}^F(X)$. Thus, this simple approximation of $\mathcal{V}_{\text{eff}}^p$ with the free T -matrix can be used also for predicting properties of nuclei nearby the drip-lines. It should be mentioned that one approximation used in this analysis looks doubtful for small values of μ . This is neglecting all the pairing effects in the complementary space. This drawback can be improved by small change of the model space by addition to it a set of states with small positive energies $\varepsilon_\lambda < E_0$ where E_0 is of the order of several MeV. Estimates show that in this case the difference between $\mathcal{V}_{\text{eff}}^p$ and the T -matrix becomes even smaller.

Thus, the free off-shell T -matrix taken at the energy $E = 2\mu$, indeed, is a good approximation for the microscopic effective pairing interaction. For the separable representation of the Paris NN -potential it can be easily found by solving eqs. (11)–(13). As far as we deal with comparatively small shift from the mass shell, all the realistic NN -potentials must give approximately the same predictions for the T -matrix.

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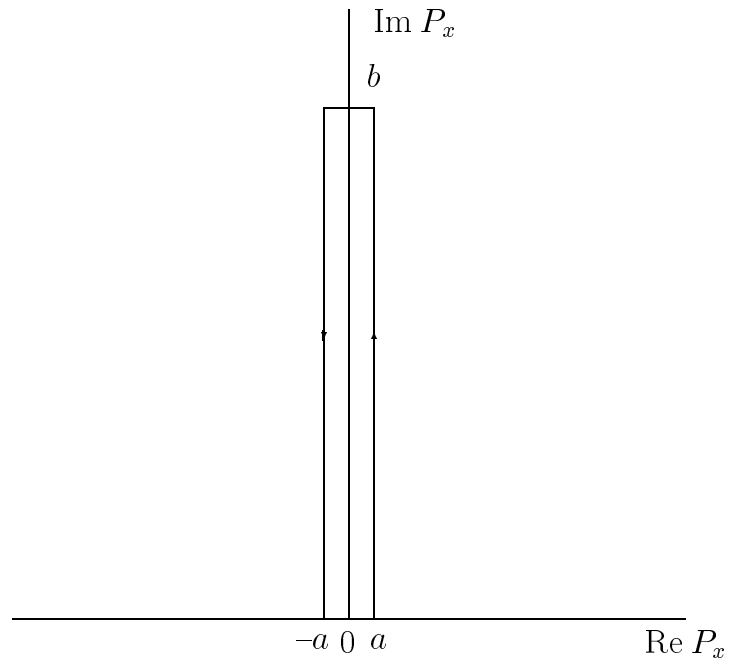


Figure 1: The integration contour for the inverse Fourier transformation (14).

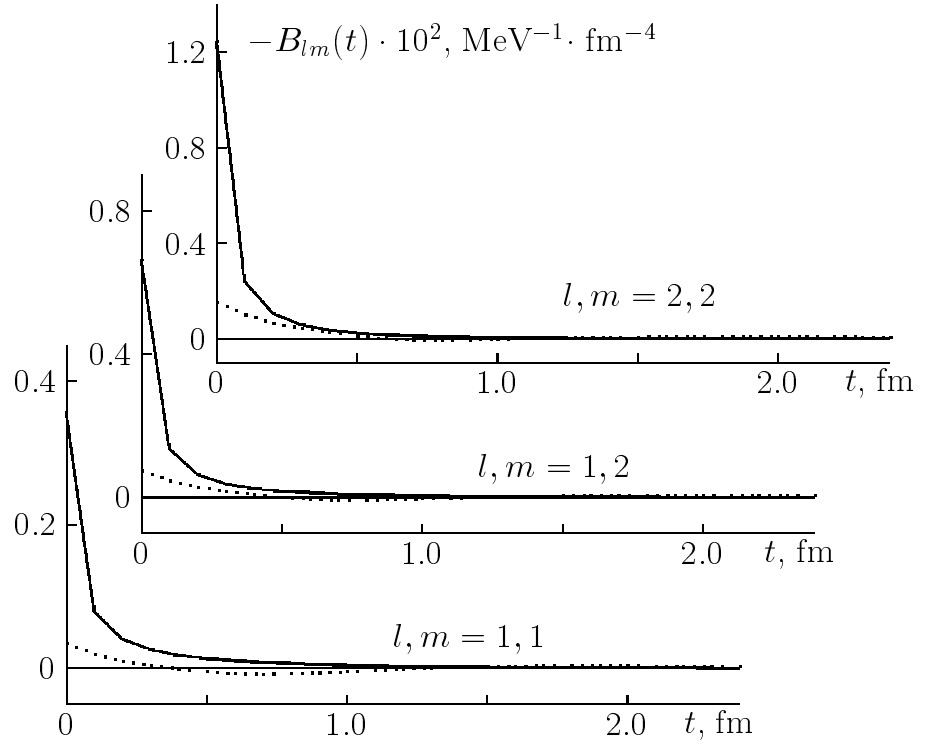


Figure 2: The free propagators $-B_{lm}^0$ (solid lines) and the differences $-(B_{lm}^{\text{inf}}[V = 50 \text{ MeV}] - B_{lm}^0)$ (dotted lines) calculated for $\mu = -8 \text{ MeV}$ versus relative coordinate $t = X - X'$.

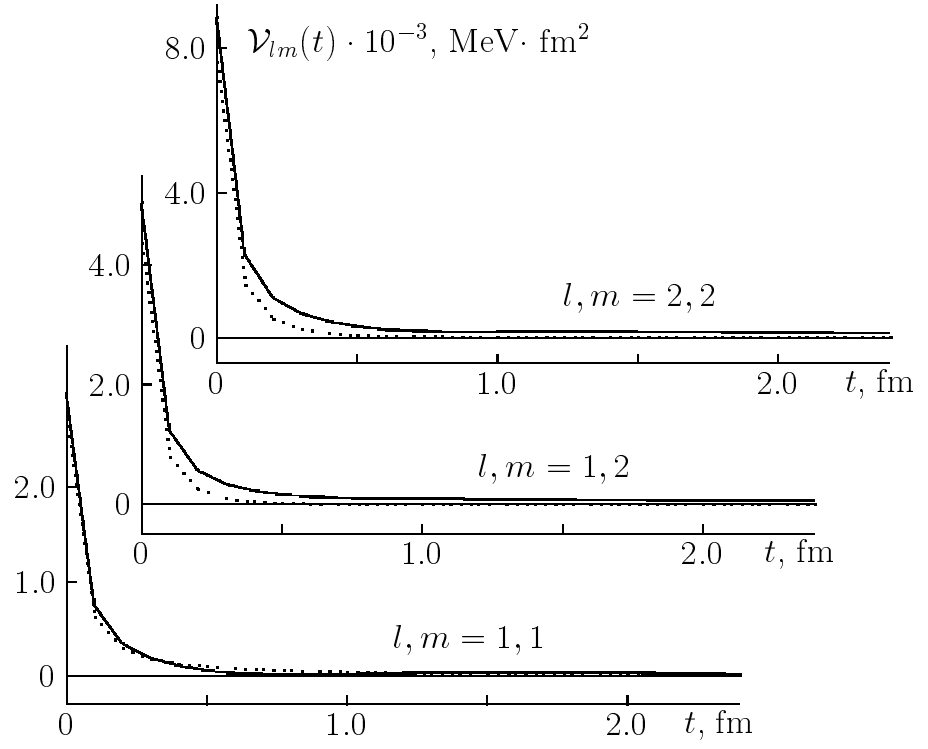


Figure 3: The lm -components of the effective interaction $\mathcal{V}_{\text{eff}}^{\text{inf}}[V = 50 \text{ MeV}]$ (solid lines) and the free T -matrix (dotted lines) calculated for $\mu = -8 \text{ MeV}$ versus t . Sign is changed for $i, j = 1, 1$ and $2, 2$.

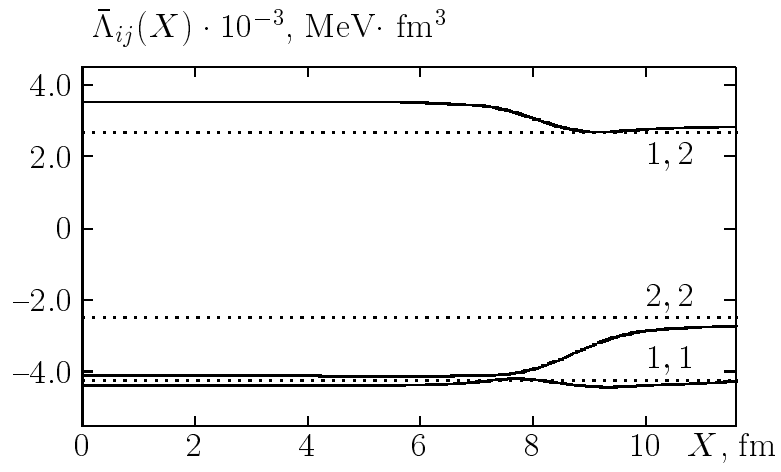


Figure 4: The zero moments $\bar{\Lambda}_{ij}(X)$ (solid lines) and the values \bar{T}_{ij} (dotted lines) for $\mu = -8 \text{ MeV}$.

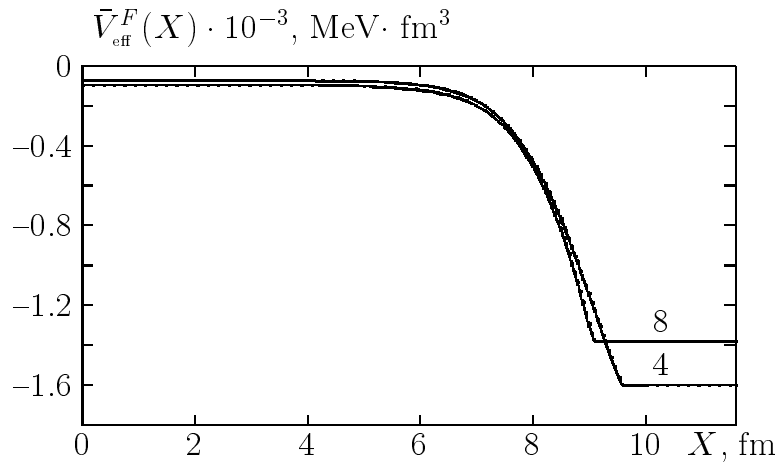


Figure 5: The effective interaction $\bar{V}_{\text{eff}}^F(X)$ (solid lines) and $\bar{T}_{ij}^F(X)$ (dotted lines) for two values of the chemical potential μ (the number corresponds to $|\mu|$ in MeV).

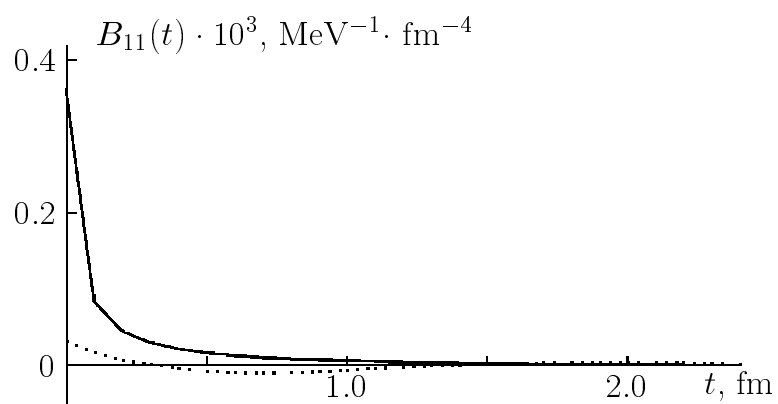


Figure 6: The same as in Fig. 2 for $l, m = 1, 1$ calculated for $\mu = -4$ MeV.